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Piecewise deterministic Markov processes and maintenance modelling: application to maintenance of a train air-conditioning system

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Abstract: This paper deals with the preventive maintenance (PM) optimization of air-conditioning systems used aboard regional trains in France by the SNCF (French Railway Company). Two kinds of PM policies are envisioned: one with a single overhaul in the whole lifetime of the air-conditioning system, another with opportunistic replacements of components that are too old at each system failure. The air-conditioning system is formed of about 20 ageing and stochastically independent components. The envisioned PM policies make them functionally dependent, however. Both PM optimizations are performed with respect to the same cost function, involving the mean number of component replacements on some finite horizon. In view of its numerical assessment, a piecewise deterministic Markov processes (PDMP) model is used, both to model the maintained and the unmaintained system; a deterministic numerical scheme is next proposed, based on finite volume (FV) methods for PDMPs; owing to difficulties in its implementation, an approximation of this scheme is next used, which is much easier to implement than the initial FV scheme. As a result of using this method, it was finally possible to optimize both PM policies, which are both proved to lower the cost function of about 7 per cent.

Keywords: multi-unit system, stochastic modelling, preventive maintenance policy, piecewise deterministic Markov processes, numerical assessment, finite volume scheme, railway

1 INTRODUCTION

For a railway company such $ as the Société Nationale des Chemins de Fer (SNCF, French Railway Company), maintenance of rolling stock constitutes a major task: a material failure is expensive and causes customer dissatisfaction. The SNCF has hence initiated research studies in order to model such systems, in view of their preventive maintenance. This article deals with an air-conditioning system.

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The air-conditioning system is a serial/parallel system consisting of 17 components. The components' lifetimes are Weibull distributed, with shape factors greater than 1, which implies that they are ageing. The objective is to optimize the air-conditioning maintenance with respect to the maintenance mean cost. Two different maintenance strategies are tested: a single overhaul and an opportunistic maintenance strategy.

Because of the components' ageing, the usual Markov processes such as Markov jump processes cannot be used. Consequently, in order to model the air-conditioning system, Markov processes called piecewise deterministic Markov processes (PDMPs) are used. Those processes are described by Davis [**1**, **2**]. Their numerical assessment is often established by Monte Carlo simulations, see references [**3**] and [**4**]; however, with this method it usually takes too much time to optimize maintenance. The present authors hence propose an alternate method: first it is observed that the quantities of interest can be expressed using PDMP marginal distributions, which are known to be solutions of a set of partial differential equations called Chapman–Kolmogorov equations. A finite volume (FV) scheme is next proposed, which provides numerical estimates for the PDMP marginal distributions, as a solution of this scheme. The memory space needed for its implementation is, however, too large in this instance of the air-conditioning system. So an approximation of the FV scheme is used, which reduces the space memory and allows the system of interest to be quantified. The results found with this method are compared with those found with Monte Carlo simulations.

This paper is organized as follows: in section 2, the air-conditioning system is presented, as well as the PDMP used to model it and the FV algorithm. In section 3, the preventive maintenance strategies are first presented and modelled with PDMPs, and the associated cost functions are provided, with respect of the PDMPs marginal distributions. The approximation method used for their numerical assessment is next presented. Results are provided in section 4 and optimal maintenance strategies are determined for the air-conditioning system. Section 5 provides some concluding remarks to end the paper.

2 THE AIR-CONDITIONING SYSTEM, MODELLING, AND QUANTIFICATION METHODS

This section is devoted to the unmaintained air-conditioning system, only submitted to corrective actions.

2.1 The air-conditioning system

Figure 1 describes the air-conditioning system. It has 17 ageing components. Some of them are in active redundancy and the others are in series. The first part has two circuits called *A* and *B*. There are five components on circuit *A* and five on circuit *B*. Circuits *A* and *B* are identical. The two branches work together. When a component on one of these branches fails, the components on the same branch stop ageing. The system crashes if one component in series fails or if one component on circuit *A* and one component on circuit *B* fail. When the air conditioning stops working, it is instantly repaired. The restoration consists in instantaneously replacing all broken components with new ones. The components are stochastically independent but the corrective maintenance strategy makes them functionally dependent. When a component of part *A* or *B* fails, it is only repaired when another component of another part of the system fails; this makes components of parts *A* and *B* dependent upon each other, and upon components of part *S* also.

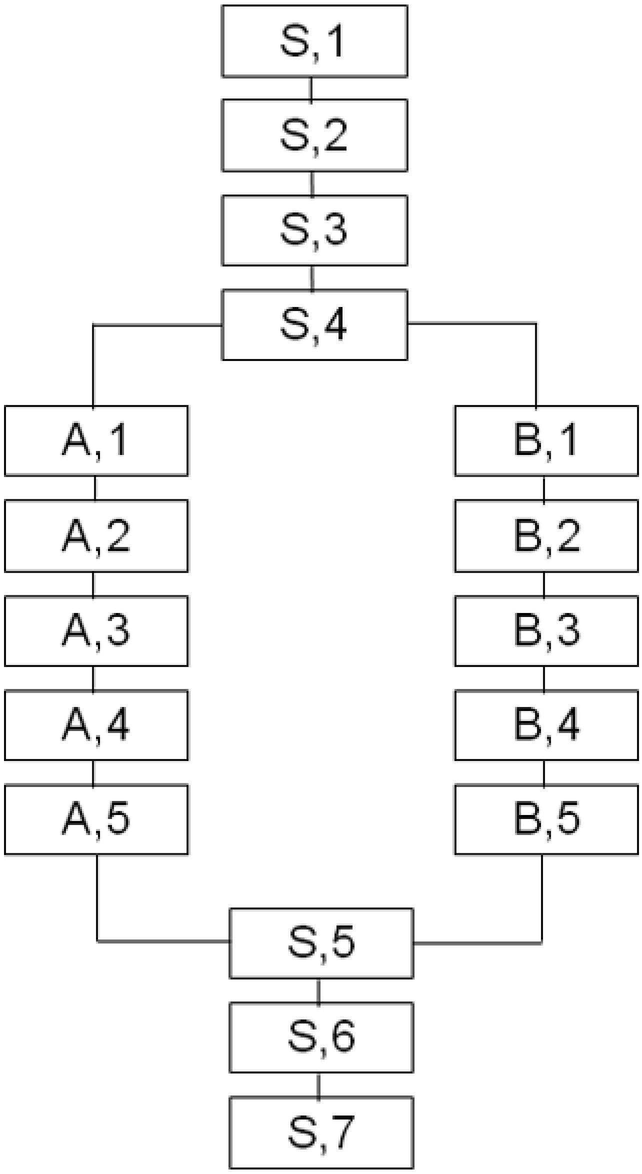

Fig. 1 Diagram of the air-conditioning system

The components lifetimes are Weibull distributed. The probability distribution function (p.d.f.) of a Weibull distribution with shape parameter denoted by β and scale parameter denoted by η is given by formula (1). The Weibull parameters and components costs are presented in Table 1, which provides fictive data, because of confidentiality problems. A complete description of the components has not been included in the paper, for the same reason. The components' failure rates are not constant because the lifetime

Table 1 Weibull distributions coefficients and costs of the components

Component	Shape	Scale	Cost (€)
S,1	1.5	30	300
S,2	2	20	400
S,3	1.5	80	1000
S,4	2.5	50	800
S,5	1.2	60	250
S,6	2	20	400
S,7	3	40	300
A-B,1	2.5	35	200
A-B,2	1.3	25	1000
A-B,3	2	50	400
A-B,4	1.5	45	300
A-B,5	1.8	20	200

distributions are Weibull's with a shape parameter higher than one.

$$f(x) = \frac{\beta}{\eta}\left(\frac{x}{\eta}\right)^{\beta-1} \mathrm{e}^{-(x/\eta)^{\beta}}, \quad \forall x \in R_{+}, \tag{1}$$

2.2 Modelling by PDMP

Piecewise deterministic Markov processes were introduced by Davis in 1984 [**1**, **2**]. This type of modelling is now used for what Devooght called 'dynamic reliability' according to the vocabulary he introduced for nuclear issues [**5**]. A PDMP is a hybrid process $(I_t, X_t)_{t\geqslant 0}$. The first component I_t is discrete, with values in a finite state space E. Typically, it indicates the state – up or down – for each component of the system at time t. The second component X_t, with values in a Borel subset $B \subset R^d$, stands for environmental conditions, such as temperature, pressure, and in the present case, the ages of components. This means that a PDMP can model a system with ageing components. The two parts I_t andX_t interact one with each other: the process jumps at countably many isolated random times; by a jump from $(I_{t}-, X_{t}-) = (\eta, x)$ to $(I_t, X_t) = (\sigma, y)$ (with $(\eta, x), (\sigma, y) \in E \times B$), the transition rate between the discrete states η and σ depends on the environmental condition x just before the jump, and is a function $x \to a(\eta, \sigma, x)$. Similarly, the environmental condition X_t just after the jump, is distributed according to some distribution $\mu_{(\eta,\sigma,x)}(\mathrm{d}y)$, which depends on both components just before the jump (η, x) and on the after jump discrete state σ. So the transition kernel which governs the transition between (η, x) and (σ, y) is $b((\eta, x); (\sigma, \mathrm{d}y)) = a(\eta, \sigma, x)\ \mu_{(\eta,\sigma,x)}(\mathrm{d}y)$. Between jumps, the discrete component I_t is constant and the evolution of the environmental condition X_t is deterministic, solution of a set of differential equations which depends on the fixed discrete state: given that $I_t = \eta$ between two jumps, X_t is solution of

$$\frac{\mathrm{d}y}{\mathrm{d}t} = v(\eta, y) \tag{2}$$

Under standard conditions, equation (2) admits a single solution such that $y(0) = x$, which is denoted by $g(\eta, x, t)$.

In case B is bounded with boundary Γ, jumps may also be induced by the reaching at Γ by $(X_t)_{t\geqslant 0}$. When X_t reaches the boundary in (η, x), the after-jump distribution is then denoted by $q((\eta, x); (\sigma, \mathrm{d}y))$.

In the case of the air-conditioning system, component I_t stands for the discrete state of the system. Setting 0 for a down component and 1 for an up one, this would lead to a state space equal to $\{0, 1\}^{17}$ (0 for down, 1 for up). Owing to the system operating mode (with instantaneous repairs), the process actually evolves in a small part E of $\{0, 1\}^{17}$, which is composed of

(a) $\mathbb{1}$: 'all the components of the system work'
(b) $\mathbb{1}_{K,i}$: 'the system works, but component (K, i) is down' with $K \in \{A, B\}$

So $E = \{\mathbb{1}, \mathbb{1}_{K,i}$ where $K \in \{A, B\}$ and $i \in \{1, \ldots, 5\}\}$.

As for the second component X_t of the PDMP, it describes the ages of all components at time t. There are 17 components and the system is modelled during $L = 30$ years, so that X_t takes the range in $[0; L]^{17}$.

As all components' lifetimes are Weibull distributed with shape parameters greater than one (see Table 1), the failure rates of the functioning components depend on their (respective) ages, which increase with time at speed 1. As for a suspended component (in a failed branch A or B), it is not ageing any more and its age evolves at speed 0.

When the system is in the perfect working state $\mathbb{1}$, all components are ageing and the solution of equation (2) hence is

$$\forall x \in R_{+}^{17}, \forall t \in R_{+},$$
$$g(\mathbb{1}, x, t) = x + t \cdot \mathbf{1}$$
$$\text{with } \mathbf{1} \text{ the unit vector with 17 components} \tag{3}$$

When the system is in state $\mathbb{1}_{K,i}$ with $K \in \{A, B\}$, all components are ageing except for those which are on branch K. This provides

$$\forall x \in R_{+}^{17}, \forall t \in R_{+}, \forall K \in \{A, B\}, \forall i \in \{1, \ldots, 5\}$$
$$g(\mathbb{1}_{K,i}, x, t) = x + t\ \varepsilon^{K,i} \tag{4}$$

with $\varepsilon^{K,i} \in \{0, 1\}^{17}$ such that $\varepsilon^{K,i}_{K,j} = 0$ for all $j \in \{1, \ldots, 5\}$ and $\varepsilon^{K,i}_{L,j} = 1$ for all other component (L, j) with $L \neq K$.

The process jumps each time a component fails. Owing to the stochastic independence of the components, the failure rate of component (K, i) only depends on its age $x_{K,i}$ and is denoted by $a_{K,i}(x_{K,i})$ with $K \in \{A, B, S\}$. The possible transitions kernels then are:

(a) in case of failure of (K, i) where $K \in \{A, B\}$ with all other components up

$$b((1, x), (1_{K,i}, \mathrm{d}y)) = a_{K,i}(x_{K,i})\delta_x(\mathrm{d}y)$$

where $\delta_x(\mathrm{d}y)$ stands for the Dirac mass at x and means that all components' ages remain unchanged;

(b) in case of failure of a component in part L when a component (K, i) is already down, with $K \in \{A, B\}$ and $L \neq K$: both down components are instantaneously replaced by new ones so that their ages are reset to zero and the other ones remain unchanged

$$b((1_{K,i}, x), (1, \mathrm{d}y)) = \sum_{\substack{(L,j) \\ L\in\{A,B,S\} \\ L\neq K}} a_{L,j}(x_{L,j})\delta_0(\mathrm{d}y_{K,i})\delta_0(\mathrm{d}y_{L,j}) \prod_{(M,k)\notin\{(K,i),(L,j)\}} \delta_{x_{M,k}}(\mathrm{d}y_{M,k})$$

(c) in case of failure of a component in part S, which is instantaneously replaced by a new one

$$b((1, x), (1, \mathrm{d}y)) = \sum_{(S,i)} a_{S,i}(x_{S,i})\delta_0(\mathrm{d}y_{S,i}) \prod_{(M,j)\neq(S,i)} \delta_{x_{M,j}}(\mathrm{d}y_{M,j})$$

This achieves the modelling of the air-conditioning system by a PDMP.

In order to optimize the air-conditioning system maintenance, the mean maintenance cost of this system under the two envisioned maintenance strategies has to be computed. This quantity may be expressed in terms of the marginal distribution of the PDMP. More specifically, let $\pi_t(\eta, \mathrm{d}x)$ stand for the distribution of (I_t, X_t) at time t. All the quantities of interest for the cost function may then be expressed with respect to $\pi_t(\eta, \mathrm{d}x)$ in shapes provided by equations (5) and (6).

$$E[f(I_t, X_t)] = \sum_{\eta\in E} \int_{R^d} f(\eta, x)\pi_t(\eta, \mathrm{d}x) \quad (5)$$

$$\int_0^t E[f(I_s, X_s)]\mathrm{d}s = \int_0^t \sum_{\eta\in E} \int_{R^d} f(\eta, x)\pi_s(\eta, \mathrm{d}x)\mathrm{d}s \quad (6)$$

For instance, the cost function involves the mean number of system failures $N_d(t)$ occurring before time t, which is given in equation (7).

$$\begin{aligned} E[N_d(t)] = & \sum_{\substack{(K,i) \\ K\in\{A,B\}}} \int_0^t \int_{[0;30]^{17}} a(1, 1_{K,i}, x)\pi_s(1, \mathrm{d}x)\mathrm{d}s \\ & + \int_0^t \int_{[0;30]^{17}} a(1, 1, x)\pi_s(1, \mathrm{d}x)\mathrm{d}s \\ & + \sum_{\substack{(K,i) \\ K\in\{A,B\}}} \int_0^t \int_{[0;30]^{17}} a(1_{K,i}, 1, x)\pi \end{aligned} \quad (7)$$

with

$$a(1, 1, x) = \sum_{(S,i)} a_{S,i}(x_{S,i}),$$

$$a(1_{K,i}, 1, x) = \sum_{\substack{(L,j) \\ L\neq K}} a_{L,j}(x_{L,j}) \quad \forall K \in \{A, B\},$$

and

$$a(1, 1_{K,i}, x) = a_{K,i}(x_{K,i})$$

There is no explicit expression for the PDMP marginal distribution $\pi_t(\eta, \mathrm{d}x)$, so it has to be numerically estimated. A FV algorithm is next presented, for that purpose.

2.3 The FV algorithm

Using the fact that a PDMP is a Markov process (with general state space), the associated Chapman–Kolmogorov equation may be written, as displayed in equation (8), see reference [**6**] for more details.

$$\begin{aligned} & \forall t \in R^+, \forall \phi \in C_c^1(E \times R^d), \\ & \int_0^t \sum_{\eta\in E} \int_{R^d} \sum_{\sigma\in E} a(\eta, x, \sigma) \\ & \times \left[\int_{R^d} \phi(\sigma, y)\mu_{(\eta,x,\sigma)}(\mathrm{d}y) - \phi(\eta, x) \right] \pi_s(\eta, \mathrm{d}x)\mathrm{d}s \\ & + \int_0^t \sum_{\eta\in E} \int_{R^d} v(\eta, x) \cdot \nabla\phi(\eta, x)\pi_s(\eta, dx)ds \\ & - \sum_{\eta\in E} \int_{R^d} \phi(\eta, x)\pi_t(\eta, \mathrm{d}x) \\ & + \sum_{\eta\in E} \int_{R^d} \phi(\eta, x)\pi_0(\eta, \mathrm{d}x) = 0 \end{aligned} \quad (8)$$

This equation represents some balance in terms of probability flows, which takes into account both of the deterministic evolution between jumps (which evolves with speed $v(\eta, x)$) and the jumps (governed by $a(\eta, x, \sigma)\mu_{(\eta,x,\sigma)}(\mathrm{d}y)$). Finite volume methods are known to be well adapted for their numerical resolution [**7–9**]. Their principle is based on the discretization of both time and environmental state spaces. The time evolution of the probability masses in each cell of the environmental state space is followed (time) step by step and, at each step, some balance is written between the out- and in-coming probability masses. The FV algorithm proposed in the current paper computes an approximation of $\pi_t(\eta, \mathrm{d}x)$ which admits a density $\bar{\pi}_t(\eta, x)$ with respect of Lebesgue measure, constant on each time step and each cell of the environmental state space.

To be more specific, a regular mesh D of $[0, L]^d$ is considered, where $[0, L]^d$ is divided into regular cells of the shape $M = [m_1 \cdot h; (m_1 + 1) \cdot h[\times \cdots \times [m_d \cdot h; (m_d + 1) \cdot h[$, with h the discretization step and $(m_1, \ldots m_d) \in N^d$. The time step is taken equal to δt and the constant value of $\bar{\pi}_t(\eta, x)$ when $n \cdot \delta t \leqslant t < (n + 1) \cdot \delta t$ and $x \in M$ is denoted by $u_n(M, i)$.

The FV algorithm is first initialized by

$$u_0(M, \eta) = \frac{1}{h^d} \int_M \pi_0(\eta, dx) \quad \forall \eta \in E \quad \forall M \in D$$

where h^d stands for the volume of the cell M and $\pi_0(\eta, \mathrm{d}x)$ for the initial distribution of the process $(I_t, X_t)_{t>0}$.

New notation needs to be introduced to write the evolution of the probability masses between step n and step $n + 1$. In this way, let

$$a^{\eta,\sigma}_{M,N} = \frac{1}{h^d} \int_M \left[\int_N b((\eta, x), (\sigma, \mathrm{d}y)) \right] \mathrm{d}x$$
$$\forall \eta, \sigma \in E^2 \quad \forall M, N \in D^2$$

stand for the discrete transition rate between the cells (η, M) and (σ, N) and let

$$\lambda^\eta_M = \sum_{j \in E} \sum_{N \in D} a^{\eta,\sigma}_{M,N} \quad \forall \eta \in E \quad \forall M \in D$$

stand for the discrete exit rate from the cell (η, M).

In the FV algorithm, the replacements make masses return to zero and the ageing makes the masses move from one cell to adjacent cells in one time step. For $M = [m_1 \cdot h; (m_1 + 1) \cdot h[\times \cdots \times [m_d \cdot h; (m_d + 1) \cdot h[$ and $N = [n_1 \cdot h; (n_1 + 1) \cdot h[\times \cdots \times [n_d \cdot h; (n_d + 1) \cdot h[$ such that $n_k = m_k + 1$ for some $k \in \{1, \ldots, d\}$ and $n_l = m_l$ for all other l, the probability mass moves from M to N and from N to M with respective speeds $v^\eta_{M,N} = 1$ and $v^\eta_{N,M} = -1$. If M and N are not adjacent, let $v^\eta_{N,M} = 0$.

Let N_M be the neighbouring cells of M. The FV algorithm is written as follows

$$\begin{aligned}
&\forall \eta \in E, \forall M \in D,\\
&u_{n+1}(M, \eta)\\
&= \left[1 - \delta t \left(\sum_{N \in N_L} \frac{1}{h} 1_{v^\eta_{M,N}=1} + \lambda^\eta_M \right) \right] u_n(M, \eta)\\
&\quad - \delta t \sum_{L \in N_M} \frac{1}{h} u_n(N, \eta) 1_{v^{\eta i}_{M,N}=-1}\\
&\quad + \delta t \sum_{\sigma \in E} \sum_{N \in D} a^{j,\eta}_{N,M} u_n(N, \sigma) \qquad (9)
\end{aligned}$$

A sufficient condition for this algorithm to be stable is that the coefficient of $u_n(M, \eta)$ is non-negative, which here is written as

$$1 - \delta t \left(\sum_{L \in N_K} \frac{1}{h} 1_{v^\eta_{M,N}=1} + \lambda^\eta_M \right) \geqslant 0 \qquad (10)$$

In all the following, the time discretization step δt is taken as the maximum value that makes the algorithm stable and is given by an equality in condition (10). The computation time and the required memory space consequently depend on one single parameter: the discretization step h of the environmental state space.

The issue with the previous FV scheme is that the memory space needed for its implementation can be important. If the discretization step of the environmental state space (h) is one, we have to keep in memory 30^{17} values per each system discrete state. To reduce that number, one idea would be to isolate independent components. However, even if the components are assumed to be stochastically independent, the corrective maintenance strategy makes them functionally dependent, as already described. No component can hence be isolated and the dimension of the environmental state space cannot be reduced in that way. An approximation must therefore be used, in order to reduce the required memory space.

2.4 Approximation of the FV algorithm

For sake of simplicity, in all the following, $u_n(M, \eta)$ is used to denote for all $M = [x_1; x_1 + h[\times \cdots \times [x_d; x_d + h[$ and $x = (x_1, \ldots, x_{17})$.

The aim of the approximation is to reduce the computations of the multidimensional $u_n(x, \eta)$ to one-dimensional functions. To do that, the cells of the mesh D are first grouped into sets of cells denoted by S (see Fig. 2 for an illustration in dimension 2).

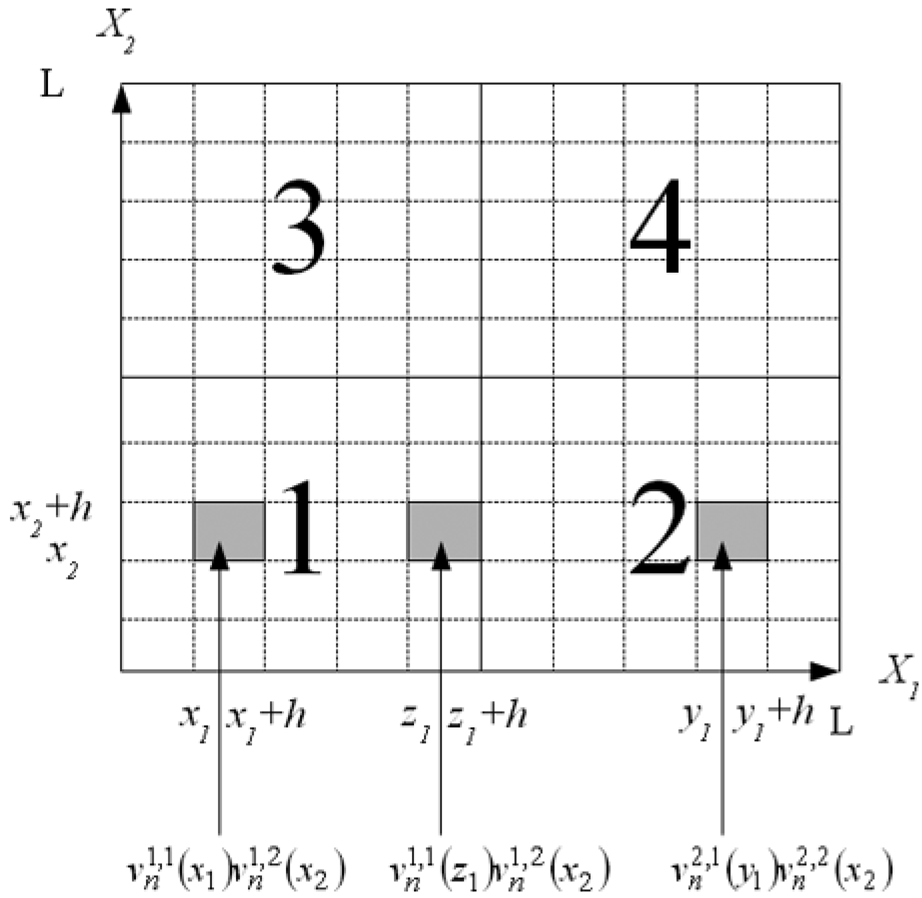

Fig. 2 Illustration of the approximation

For each cell $M = [x_1; x_1 + h[\times \cdots \times [x_d; x_d + h[$ in S and each η in E, $u_n(x, \eta)$ is then written as the products of functions which depend on S, see equation (11) just below. Alternatively, $u_n(x, \eta)$ can be written as a sum of functions which all are zeros except from a single one, which corresponds to the group S, see equation (12).

$$u_n(x, \eta) \approx \prod_{j=1}^{d} v_n^{C,j}(x_j, \eta), \quad \forall \eta \in E,\ \forall M \in C \text{ with } C \in S \tag{11}$$

$$u_n(x, \eta) \approx \sum_{G} \prod_{j=1}^{d} v_n^{G,j}(x_j, \eta), \forall \eta \in E, \quad \forall M \in D, \tag{12}$$

where $v_n^{G,j}(x_j, \eta) = 0$ if $M \notin G$.

To better understand the method, it is now described in the bi-dimensional case with a single discrete state. This discrete state is hence omitted and the point is to see how to approximate $u_n(x) = u_n(x_1, x_2)$. The physical variable is assumed to evolve in space $[0; L]^2$.

The cells of the mesh are here grouped into four groups G, $G \in \{1, 2, 3, 4\}$. These can be seen in Fig. 2. On each of those groups, the approximation consists in writing $u_n(x_1, x_2)$ as the product of two functions dependent on G, one per component.

If $x = (x_1, x_2)$ is in group G, $G \in \{1, 2, 3, 4\}$

$$u_n(x_1, x_2) \approx v_n^{G,1}(x_1) v_n^{G,2}(x_2), \quad \forall (x_1, x_2) \in G \tag{13a}$$

If $x = (x_1, x_2)$ is not in group G, $v_n^{G,1}(x_1) v_n^{G,2}(x_2) = 0$.

Equation (12) also can be written as

$$u_n(x_1, x_2) \approx \sum_{G=1}^{4} v_n^{G,1}(x_1) v_n^{G,2}(x_2), \quad \forall (x_1, x_2) \in D, \tag{13b}$$

The algorithm (9) may be written in a simpler form

$$u_{n+1}(x_1, x_2) = \sum_{(y_1, y_2) \in D} b[(y_1, y_2), (x_1, x_2)] u_n(y_1, y_2), \quad \forall (x_1, x_2) \in D, \tag{14}$$

Substituting equation (13b) into (14) provides

$$v_{n+1}^{G,1}(x_1)v_{n+1}^{G,2}(x_2) = \sum_{H=1}^{4}\sum_{(y_1,y_2)\in H} b[(y_1,y_2),(x_1,x_2)]v_n^{H,1}(y_1)v_n^{H,2}(y_2),$$
$$\forall (x_1,x_2)\in G, \forall G\in\{1,2,3,4\} \quad (15)$$

A way to calculate $v_n^{H,1}(y_1)$ and $v_n^{H,2}(y_2)$ has to be found. Group G is defined as the product of two sets $I_1^G \times I_2^G$. Let

$$C_n^{G,1} = \sum_{x_1\in I_1^G} v_n^{G,1}(x_1) =$$
$$\text{and } C_n^{G,2} = \sum_{x_2\in I_2^G} v_n^{G,2}(x_2) =, \quad \forall G\in\{1,2,3,4\} \quad (16)$$

Summing equation (15) over x_2 in I_2^G gives

$$v_{n+1}^{G,1}(x_1)C_{n+1}^{G,2} = \sum_{x_2\in I_2^G}\sum_{H=1}^{4}\sum_{(y_1,y_2)\in H} b[(y_1,y_2),(x_1,x_2)]v_n^{H,1}(y_1)$$
$$v_n^{H,2}(y_2), \forall x_1\in I_1^G, \quad (17)$$

Let

$$f_n^{G,1}(x_1) = v_n^{G,1}(x_1)C_n^{G,2} \text{ and } C_n^G = C_n^{G,1}\cdot C_n^{G,2} \quad (18)$$

Equation (17) may now be written as

$$f_{n+1}^{G,1}(x_1) = \sum_{x_2\in I_2^G}\sum_{H=1}^{4}\sum_{(y_1,y_2)\in D} b[(y_1,y_2),(x_1,x_2)] \times \frac{f_n^{H,1}(y_1)f_n^{H,2}(y_2)}{C_n^G}, \quad \forall x_1\in I_1^G \quad (19)$$

Also, summing over x_1 in I_1^G in the left equality of (18) provides

$$C_n^G = \sum_{x_1\in I_1^G} f_n^{G,1}(x_1), \quad \forall x_1\in I_1^G \quad (20)$$

The functions $f_n^{G,1}$ and C_n^G can now be iteratively computed using equations (19) and (20).

A similar method may be used with as many groups as wanted and in higher dimensions too. The larger the number of groups used is, the more accurate the approximation will be. For the air-conditioning system, the approximation consists in writing the estimation of the PDMP marginal distributions as the product of 17 functions for each state. The approximation simplest case is used in the following, which consists in taking one single group of cells. The approximation then becomes

$$u_n(x,\eta) \approx \prod_{j=1}^{17} v_n^j(x_j, i), \quad \forall M\in D,$$
$$\text{with } M = [x_1; x_1+h[\times\cdots\times[x_{17}; x_{17}+h[\quad (21)$$

Thanks to the previous method, the quantities useful for the computation of the maintenance mean cost can be estimated. The preventive maintenance policies are presented in the next section.

3 THE PREVENTIVE MAINTENANCE POLICIES FOR THE AIR-CONDITIONING SYSTEM

3.1 The preventive maintenance strategies

Two types of maintenance are envisioned as described below.

Case 1 The first one is based on a single overhaul, and during this review, the components which are older than a specific limit age are replaced, in addition to the broken components. Such overhauls are classically used in industry and are used by the SNCF for the maintenance of air-conditioning systems. A major interest for such a PM is that it is planned in advance, which allows repair-men and spare components to be prepared beforehand. Typically, in such an overhaul, only broken and overly degraded components are changed, where the degradation threshold for a degraded component is here modelled through a control limit age. To optimize this strategy for the air-conditioning system, 13 parameters have to be determined: one is the time at which the overhaul is executed, and the other ones correspond to the components' limit ages. In the present instance there are only 12 limit ages to find because the five limits on part *A* are the same as the limits on part *B*.

Case 2 The second one is an opportunistic maintenance strategy: the point is to take advantage of a system failure for simultaneously changing overly degraded components, in addition to the broken components. Here again, the degradation threshold is modelled through a control limit age. The main interest of such a strategy is that preventive maintenance actions are performed at the same time as corrective ones. This allows time and money to be saved, for the system need not be stopped specifically for PM actions and the repair staff need not attend specifically for them either. To optimize this strategy for the air-conditioning system, 12 components' limit ages have to be determined, just as for the case of a single overhaul.

3.2 Modelling of the preventive maintenance strategies

Both envisioned PM policies can be modelled with a PDMP, with a slight modification of the initial PDMP presented in subsection 2.2.

Case 1 If the overhaul is executed at time T, a new physical variable must be created that represents the time since entry into service. Reaching T for this new variable entails some system change of states and T then acts as a boundary for this new variable. Note that once T is reached, it is no longer needed. At time T, the components' ages are reset to zero if they are above their limit age denoted by $T(K,i)$ for $K \in \{A,B,S\}$. The transition kernels due to the transitions induced by the reaching of time T are

$$q[(1,x,T);(1,\mathrm{d}y)] = \prod_{K,i} [\delta_0(\mathrm{d}y_{K,i}) 1_{x_{K,i} \geqslant T_{K,i}} + \delta_{x_{K,i}}(\mathrm{d}y_{K,i}) 1_{x_{K,i} < T_{K,i}}] \tag{22}$$

$$q[(1_{K,i},x,T);(1,\mathrm{d}y)] = \delta_0(\mathrm{d}y_{K,i}) \prod_{(L,j)\neq(K,i)} \begin{pmatrix} \delta_0(\mathrm{d}y_{L,j}) 1_{x_{L,j} \geqslant T_{L,j}} + \\ \delta_{x_{L,j}}(\mathrm{d}y_{L,j}) 1_{x_{L,j} < T_{L,j}} \end{pmatrix}, \quad \forall K \in \{A,B\}, \tag{23}$$

Case 2 To model the opportunistic maintenance policy, there is no need to create a new physical variable. Again, let $T(K,i)$ be the limit age of the component (K,i). Equation (24) provides the transition kernel of the process when a component of branch A or B is already down when the system crashes. Equation (25) provides it when a component of part S causes the system to crash with all other components up.

$$\forall K \in \{A,B\},$$
$$b[(1_{K,i},x),(1,\mathrm{d}y)] = \sum_{\substack{(L,j)\\ L\in\{A,B,S\}\\ L\neq K}} a_{L,j}(x_{L,j})\delta_0(\mathrm{d}y_{K,i})\delta_0(\mathrm{d}y_{L,j}) \prod_{(M,K)\notin\{(K,i),(L,j)\}} \begin{pmatrix} \delta_0(\mathrm{d}y_{M,k}) 1_{x_{M,k} \geqslant T_{M,k}} + \\ \delta_{x_{M,k}}(dy_{M,k}) 1_{x_{M,k} < T_{M,k}} \end{pmatrix} \tag{24}$$

$$b[(1,x),(1,\mathrm{d}y)] = \sum_{(S,i)} a_{S,i}(x_{S,i})\delta_0(\mathrm{d}y_{S,i}) \prod_{(K,j)\neq(S,i)} [\delta_0(\mathrm{d}y_{K,j}) 1_{x_{K,j} \geqslant T_{K,j}} + \delta_{x_{K,j}}(\mathrm{d}y_{K,j}) 1_{x_{K,j} < T_{K,j}}] \tag{25}$$

Table 2 Cost of maintenance

Overhaul cost (€)	Corrective maintenance cost (€)
500	2000

To optimize the system maintenance, the overhaul time and the limit ages of the components that minimize the maintenance mean cost should now be found for both strategies.

3.3 Maintenance optimization

Table 2 provides the costs of the overhaul and corrective maintenance. When the overhaul or a corrective maintenance occurs, it costs respectively 500€ and 2000€ in addition to thereplaced components costs, see Table 1.

The objective is to find the maintenance strategy that minimizes the maintenance mean cost of the system over 30 years, which includes the following costs:

(a) C_d: system failure cost, see Table 2;
(b) $C_{K,i}$: replacement cost of component (K,i), with $K \in \{A,B,S\}$, see Table 1;
(c) C_o: overhaul cost, see Table 2 (for the opportunistic maintenance strategy, there is no overhaul so $C_o = 0$).

Other notation is also used in the following:

(a) $N_d(t)$: mean number of system failures occurred before t;
(b) $N_{K,i}(t)$: mean number of component (K,i) replacements occurred before t, with $K \in \{A,B,S\}$.

The cost function is then given by

$$C(t) = C_d\, N_d(t) + \sum_{i=1}^{7} C_{S,i}\, N_{S,i}(t) + \sum_{i=1}^{5} [C_{A,i}\, N_{A,i}(t) + C_{B,i}\, N_{B,i}(t)] + C_o \tag{26}$$

Because of the number of parameters to be optimized (12 and 13), it is not possible to test all the possibilities and a simulated annealing algorithm is used, see reference [**10**]. Each possibility is computed using the approximation of the FV algorithm (with one single group G).

4 THE RESULTS FOR THE AIR-CONDITIONONG SYSTEM

For the computations, the discretization step of the environmental state space, h, is taken as equal to one month (1/12). Table 3 shows the component optimal replacement limit ages. To find these strategies, the simulated annealing algorithm had to do between 200 and 300 tests. To accelerate the calculations, a discretization step of the environmental variable space state equal to four months (1/3) is first used, and next, when the algorithm approaches the solution, the step is switched to one month (1/12).

Table 3 Optimal strategies to minimize the mean cost

	Limit ages of replacement	
Component	Preventive maintenance at 16 years	Opportunistic maintenance
S,1	6	16
S,2	4	8
S,3		
S,4	Do not replace	Do not replace
S,5		
S,6	5	8
S,7	11	16
A-B,1	14	16
A-B,2		
A-B,3	Do not replace	Do not replace
A-B,4		
A-B,5	6	13

Table 4 Results of the mean cost optimization

Maintenance strategy	Mean cost (€)	Mean number of failures
Without preventive or opportunistic maintenance	17 293	6.4
Preventive maintenance	16 029 (−7.3%)	4.97 (−22.3%)
Opportunistic maintenance	16 064 (−7.1%)	5.1 (−20.3%)

With these strategies, the maintenance mean cost is reduced by about 7 per cent and the mean number of failures is reduced by about 20 per cent, see Table 4. Both strategies here lead to roughly the same number of failures and the same cost.

It is useful for industrial practice to compare these different maintenance strategies. An interesting quantity is the Vesely failure rate of the system defined by (27), see reference [**11**]. Figure 3 represents an approximation of the Vesely failure rate of the air-conditioning system computed with the FV algorithm, equation (28).

$$\lambda_V(t) = \lim_{\Delta \to 0} \frac{1}{\Delta} P[N_d(t+\Delta) - N_d(t) \geqslant 1], \quad \forall t \geqslant 0 \tag{27}$$

$$\bar{\lambda}_V(t^*) = \frac{1}{\delta t} P[N_d(t^* + \delta t) - N_d(t^*) = 1], \quad \forall t^* \in \{0, \delta t, 2\delta t, \ldots, \} \tag{28}$$

The effect of the different maintenance strategies can be seen in Fig. 3. After 16 years of operation, the effect of the overhaul can be observed. These results may now be compared with Monte Carlo simulations.

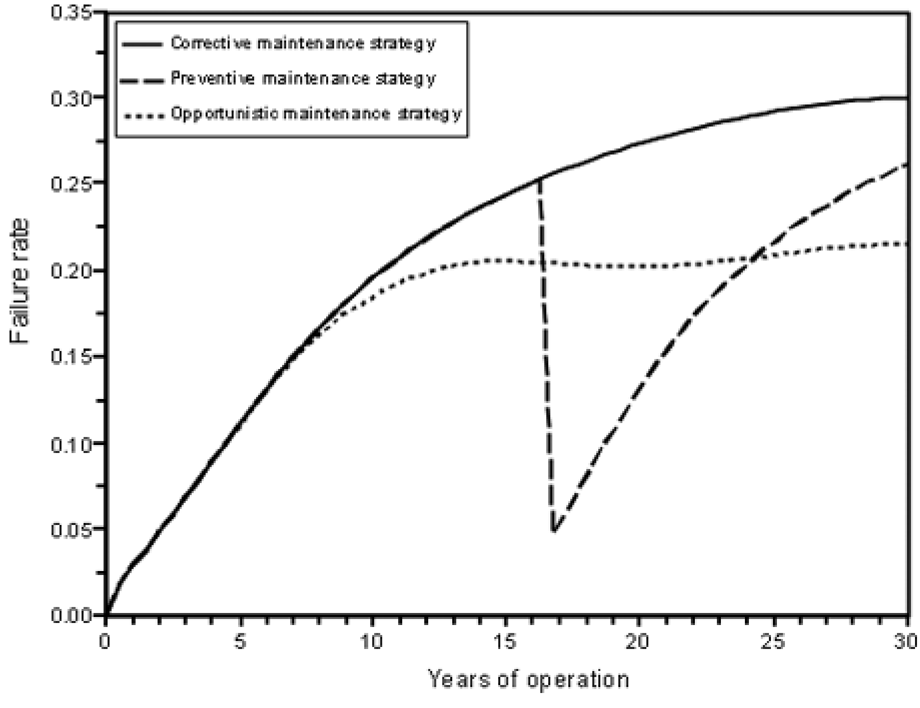

Fig. 3 Approximation of the Vesely failure rate of the air-conditioning system over three maintenance strategies

Table 5 Mean cost and mean number of system failures under corrective maintenance strategy (95 per cent CI means 95 per cent confidence interval)

Method	Mean cost (€)	Mean number of failures	CPU time
Monte Carlo (10^5 histories)	17 166 95% CI = [17 136; 17 196]	6.35 95% CI = [6.339; 6.361]	221
Finite volume ($h = 1/12$)	17 293	6.4	25
Finite volume ($h = 1/3$)	17 492	6.48	3

Table 6 Mean cost and mean number of system failures under the optimal overhaul, (95 per cent CI means 95 per cent confidence interval)

Method	Mean cost (€)	Mean number of failures	CPU time
Monte Carlo (10^5 histories)	16002 95% CI = [15 973; 16 032]	4.902 95% CI = [4.89; 4.913]	725
Finite volume ($h = 1/12$)	15992	4.97	24
Finite volume ($h = 1/3$)	16164	5.03	4

Table 7 Mean cost and mean number of system failures under the optimal opportunistic maintenance strategy, (95 per cent CI means 95 per cent confidence interval)

Method	Mean cost (€)	Mean number of failures	CPU time
Monte Carlo (10^5 histories)	15 772 95% CI = [15 743; 15 800]	4.824 95% CI = [4.814; 4.835]	1342
Finite volume ($h = 1/12$)	16 064	5.1	51
Finite volume ($h = 1/3$)	16 165	5.12	13

4.1 Precision of the results

There are two parameters which influence the precision of the results. The first parameter is the discretization step of the environmental variable state space h and the second is the number of sets used for the approximation. For this study, the second parameter is fixed to one. Two different values of h are tested: one month (1/12) and four months (1/3).

In order to verify the results found with the FV algorithm, they are compared to those found with Monte Carlo simulations. 10^5 histories are simulated. First, mean cost and mean number of failures of the air-conditioning system without preventive and opportunistic maintenance are computed with both methods. Table 5 shows the results found with these two methods. They are similar, so the results found with the FV algorithm are validated. Computations with the FV algorithm and a discretization step of 1/3 are executed in 3 s; it is fast enough to use the FV algorithm to optimize the maintenance.

In Table 6, mean cost and mean number of failures associated with the optimal overhaul are verified. The results found with the FV algorithm ($h = 1/12$ and $h = 1/3$) and those found with Monte Carlo simulations (10^5 histories) are compared. They are fairly close, so in this case the approximation used in the FV algorithm gives precise results.

Table 7 is the same as Table 6, but the results are associated with the optimal opportunistic maintenance strategy. In this case, the results are less accurate than with the overhaul. A possible explanation is that the opportunistic maintenance strategy causes a greater dependence between the component ages. The approximation used with only one group assumes some kind of independence, which is not the case. So, in order to have more precise results, a greater number of sets for the approximation should be used.

5 CONCLUSIONS

To conclude, PDMP allows the modelling of complex systems with ageing components under different maintenance strategies. A computation method has been proposed which permits the numerical assessment of reliability quantities. This method appears to be well adapted to the optimization of maintenance strategies. For the air-conditioning system, an optimal overhaul and an optimal opportunistic maintenance strategy have been determined. This methodology can be used for many systems. However, limitations are the number of ageing components and the system complexity. In future, The authors will try to apply this method for more complex systems.

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APPENDIX

Notation

$a(\eta, \sigma, x)$	PDMP transition rate between the discrete states η and σ when the environmental condition is x
d	dimension of the environmental state space
D	regular mesh of R^d
FV	finite volume
h^d	volume of cell M
I_t	PDMP discrete part at time t
p.d.f.	probability distribution function
$q((\eta, x); (., dy))$	PDMP after-jump distribution starting from (η, x) in the boundary Γ
M	cell of mesh D
PDMP	piecewise deterministic Markov process
PM	preventive maintenance
SNCF	Société Nationale des Chemins de Fer (French National Railway Society)
β	Weibull shape parameter
η	Weibull scale parameter
Γ	boundary of the PDMP state space
δt	time discretisation step for FV
$\delta_x(\mathrm{d}y)$	Dirac distribution at x
$\mu_{(\eta,\sigma,x)}(\mathrm{d}y)$	distribution of the after-jump location for X_t by a jump of I_t from η to σ, when the environmental condition before the jump is x
$\pi_t(., \mathrm{d}x)$	PDMP distribution at time t